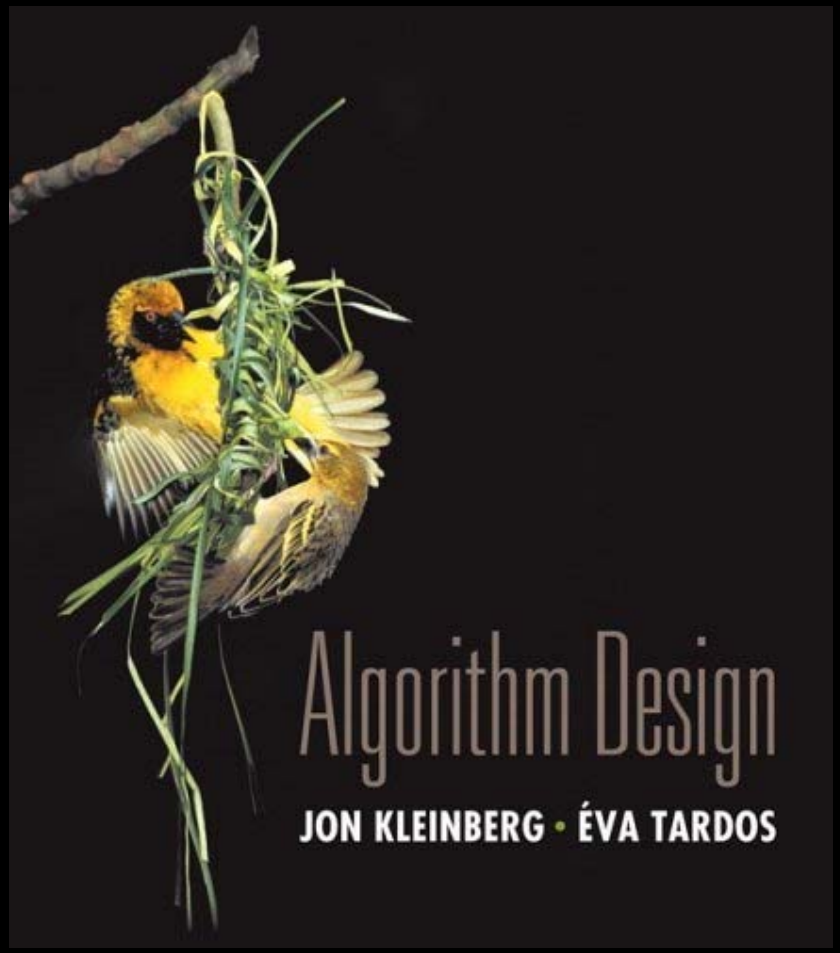


Chapter 8

NP and Computational Intractability

8.3 Definition of NP

Decision Problems

Decision problem.

- X is a set of strings.
- Instance: string s.
- Algorithm A solves problem X: A(s) = `yes` iff $s \in X$.

Polynomial time. Algorithm A runs in poly-time if for every string s, A(s) terminates in at most p(|s|) "steps", where p(·) is some polynomial.

length of s

PRIMES: X = { 2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, …. }

Algorithm. [Agrawal-Kayal-Saxena, 2002] $p(|s|) = |s|^8$.

Definition of P

P. Decision problems for which there is a poly-time algorithm.

Problem	Description	Algorithm	Yes	No
MULTIPLE	Is x a multiple of y?	Grade school division	51, 17	51, 16
RELPRIME	Are x and y relatively prime?	Euclid (300 BCE)	34, 39	34, 51
PRIMES	Is x prime?	AKS (2002)	53	51
EDIT-DISTANCE	Is the edit distance between x and y less than 5?	Dynamic programming	niether neither	acgggt ttttta
LSOLVE	Is there a vector x that satisfies Ax = b?	Gauss-Edmonds elimination	$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

NP

Certification algorithm intuition.

- Certifier views things from "managerial" viewpoint.
- Certifier doesn't determine whether $s \in X$ on its own; rather, it checks a proposed proof t that $s \in X$.

Def. Algorithm $C(s, t)$ is a certifier for problem X if for every string s, $s \in X$ iff there exists a string t such that $C(s, t) =$ `yes`.

"certificate" or "witness"

NP. Decision problems for which there exists a poly-time certifier.

$C(s, t)$ is a poly-time algorithm and $|t| \leq p(|s|)$ for some polynomial $p(\cdot)$.

Remark. NP stands for nondeterministic polynomial-time.

Certifiers and Certificates: Composite

COMPOSITES. Given an integer s, is s composite?

Certificate. A nontrivial factor t of s. Note that such a certificate exists iff s is composite. Moreover $|t| \leq |s|$.

Certifier.

```
boolean C(s, t) {
   if (t ≤ 1 or t ≥ s)
      return false
   else if (s is a multiple of t)
      return true
   else
      return false
}
```

Instance. s = 437,669.

Certificate. t = 541 or 809. ⟵ $437{,}669 = 541 \times 809$

Conclusion. COMPOSITES is in NP.

Certifiers and Certificates: 3-Satisfiability

SAT. Given a CNF formula Φ, is there a satisfying assignment?

Certificate. An assignment of truth values to the n boolean variables.

Certifier. Check that each clause in Φ has at least one true literal.

Ex.

$$(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (\overline{x_1} \vee \overline{x_3} \vee \overline{x_4})$$

instance s

$$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$$

certificate t

Conclusion. SAT is in NP.

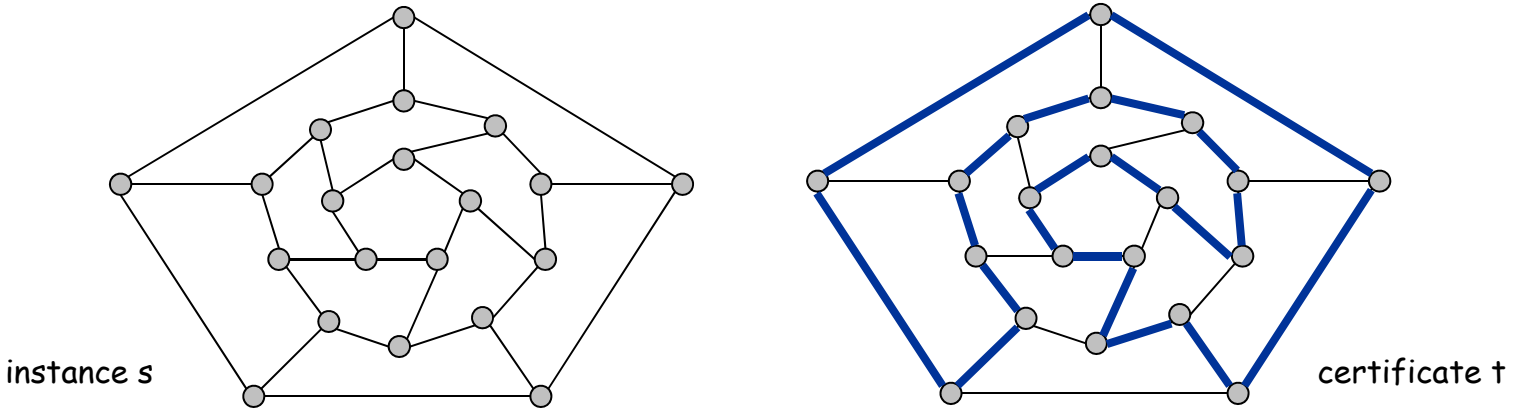
Certifiers and Certificates: Hamiltonian Cycle

HAM-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a simple cycle C that visits every node?

Certificate. A permutation of the n nodes.

Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.

Conclusion. HAM-CYCLE is in NP.



P, NP, EXP

P. Decision problems for which there is a poly-time algorithm.
EXP. Decision problems for which there is an exponential-time algorithm, $2^{p(|s|)}$.
NP. Decision problems for which there is a poly-time certifier.

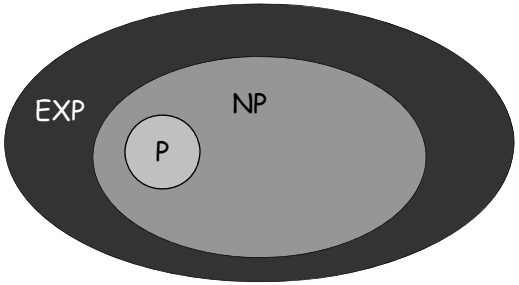
Claim. P $\subseteq$ NP.
Pf. Consider any problem X in P.
- By definition, there exists a poly-time algorithm $A(s)$ that solves X.
- Certificate: $t = \varepsilon$, certifier $C(s, t) = A(s)$. ▪

Claim. NP $\subseteq$ EXP.
Pf. Consider any problem X in NP.
- By definition, there exists a poly-time certifier $C(s, t)$ for X.
- To solve input s, run $C(s, t)$ on all strings t with $|t| \leq p(|s|)$.
- Return `yes`, if $C(s, t)$ returns `yes` for any of these. ▪

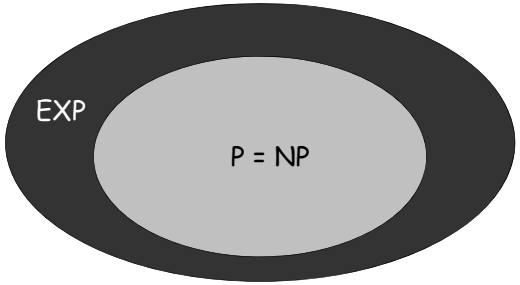
The Main Question: P Versus NP

Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]

- Is the decision problem as easy as the certification problem?
- Clay $1 million prize.



If $P \neq NP$



If P = NP

would break RSA cryptography
(and potentially collapse economy)

↙

If yes: Efficient algorithms for 3-COLOR, TSP, FACTOR, SAT, ...

If no: No efficient algorithms possible for 3-COLOR, TSP, SAT, ...

Consensus opinion on P = NP? Probably no.

The Simpson's: P = NP?

Futurama: P = NP?

Looking for a Job?

Some writers for the Simpsons and Futurama.

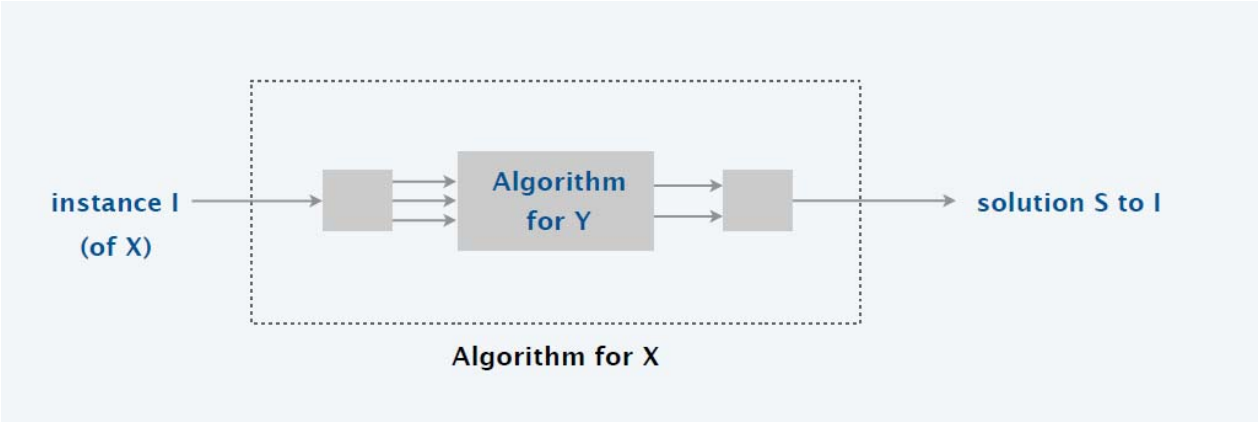
- J. Steward Burns. M.S. in mathematics, Berkeley, 1993.
- David X. Cohen. M.S. in computer science, Berkeley, 1992.
- Al Jean. B.S. in mathematics, Harvard, 1981.
- Ken Keeler. Ph.D. in applied mathematics, Harvard, 1990.
- Jeff Westbrook. Ph.D. in computer science, Princeton, 1989.

8.4 NP-Completeness

Polynomial Transformation

Def. Problem X polynomial reduces (Cook) to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.



NP-Complete

NP-complete. A problem Y in NP with the property that for every problem X in NP, $X \leq_p Y$.

Theorem. Suppose Y is an NP-complete problem. Then Y is solvable in poly-time iff P = NP.

Pf. $\Leftarrow$ If P = NP then Y can be solved in poly-time since Y is in NP.

Pf. $\Rightarrow$ Suppose Y can be solved in poly-time.

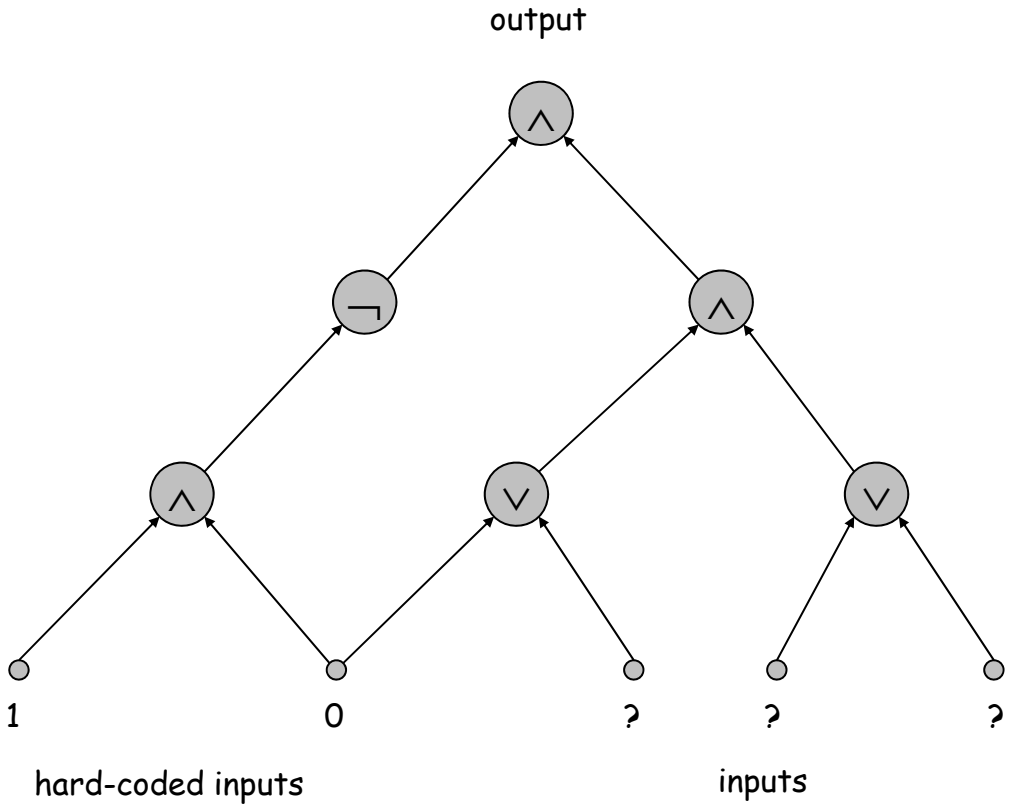
- Let X be any problem in NP. Since $X \leq_p Y$, we can solve X in poly-time. This implies NP $\subseteq$ P.
- We already know P $\subseteq$ NP. Thus P = NP. ▪

Fundamental question. Do there exist "natural" NP-complete problems?

Circuit Satisfiability

CIRCUIT-SAT. Given a combinational circuit built out of AND, OR, and NOT gates, is there a way to set the circuit inputs so that the output is 1?

yes: 1 0 1



The "First" NP-Complete Problem

Theorem. CIRCUIT-SAT is NP-complete. [Cook 1971, Levin 1973]

Pf. (sketch)

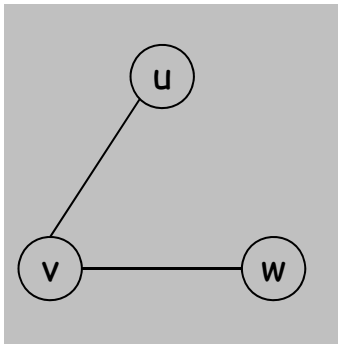
- Any algorithm that takes a fixed number of bits n as input and produces a yes/no answer can be represented by such a circuit. Moreover, if algorithm takes poly-time, then circuit is of poly-size.

  sketchy part of proof; fixing the number of bits is important, and reflects basic distinction between algorithms and circuits

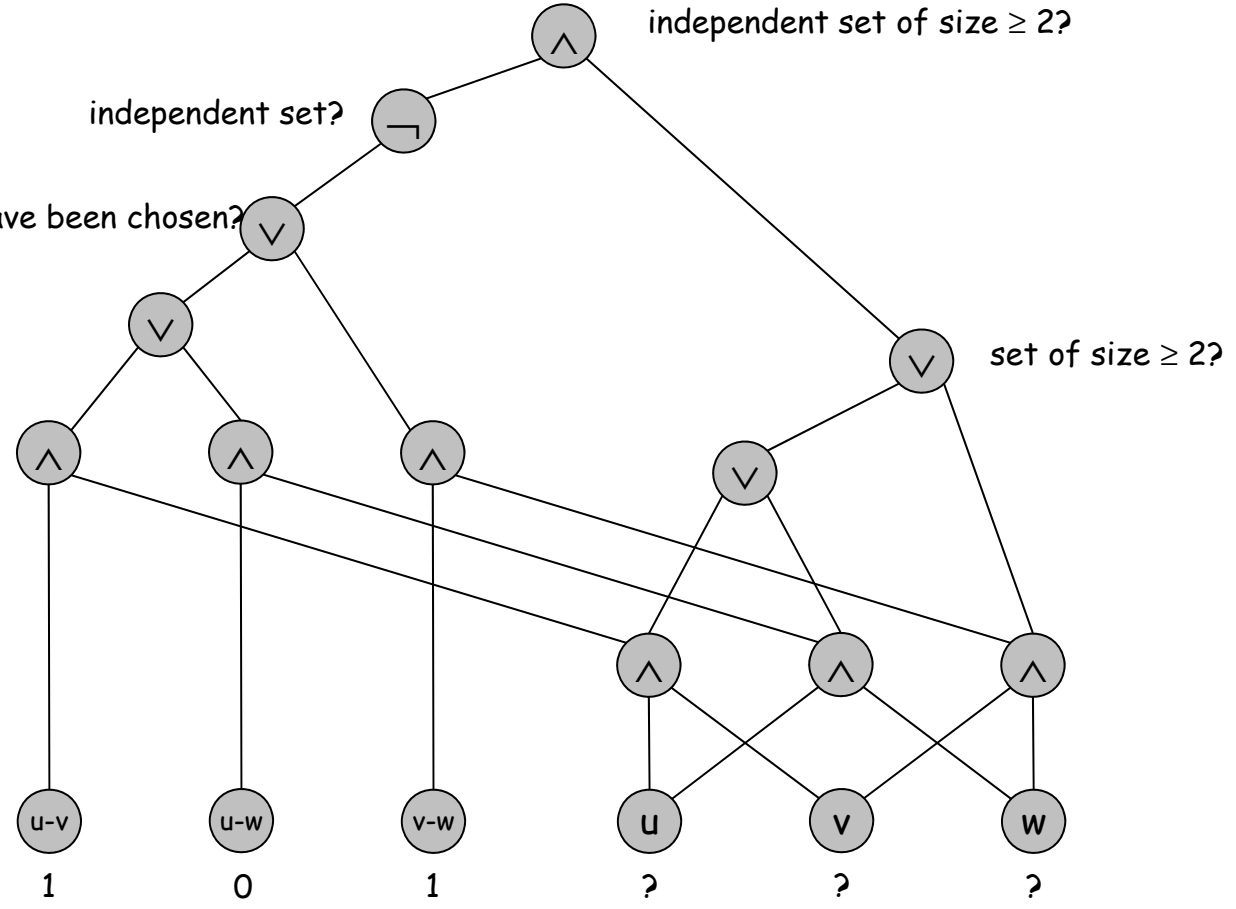
- Consider some problem X in NP. It has a poly-time certifier $C(s, t)$. To determine whether s is in X, need to know if there exists a certificate t of length $p(|s|)$ such that $C(s, t) =$ `yes`.
- View $C(s, t)$ as an algorithm on $|s| + p(|s|)$ bits (input s, certificate t) and convert it into a poly-size circuit K.
  - first $|s|$ bits are hard-coded with s
  - remaining $p(|s|)$ bits represent bits of t
- Circuit K is satisfiable iff $C(s, t) =$ `yes`.

Example

Ex. Construction below creates a circuit K whose inputs can be set so that K outputs true iff graph G has an independent set of size ≥ 2.



G = (V, E), n = 3



$\binom{n}{2}$ hard-coded inputs (graph description)

n inputs (nodes in independent set)

Establishing NP-Completeness

Remark. Once we establish first "natural" NP-complete problem, others fall like dominoes.

Recipe to establish NP-completeness of problem Y.

- Step 1. Show that Y is in NP.
- Step 2. Choose an NP-complete problem X.
- Step 3. Prove that $X \leq_P Y$.

Justification. If X is an NP-complete problem, and Y is a problem in NP with the property that $X \leq_P Y$ then Y is NP-complete.

Pf. Let W be any problem in NP. Then $W \leq_P X \leq_P Y$.

- By transitivity, $W \leq_P Y$.
- Hence Y is NP-complete. ▪

by definition of NP-complete

by assumption

3-SAT is NP-Complete

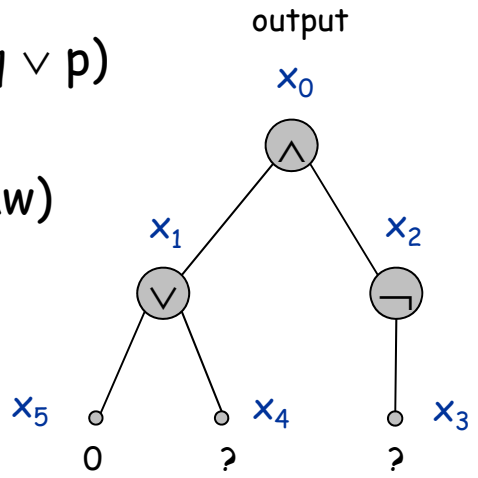
Theorem. 3-SAT is NP-complete.

Pf. Suffices to show that CIRCUIT-SAT $\leq_P$ 3-SAT since 3-SAT is in NP.

- Let K be any circuit.
- Create a 3-SAT variable x_i for each circuit element i.
- Make circuit compute correct values at each node:
  - $x_2 = \neg x_3 \Rightarrow$ add 2 clauses: $x_2 \vee x_3$, $\overline{x_2} \vee \overline{x_3}$
  - $x_1 = x_4 \vee x_5 \Rightarrow$ add 3 clauses: $x_1 \vee \overline{x_4}$, $x_1 \vee \overline{x_5}$, $\overline{x_1} \vee x_4 \vee x_5$
  - $x_0 = x_1 \wedge x_2 \Rightarrow$ add 3 clauses: $\overline{x_0} \vee x_1$, $\overline{x_0} \vee x_2$, $x_0 \vee \overline{x_1} \vee \overline{x_2}$

  - $p \rightarrow q \Leftrightarrow \neg p \vee q$
  - $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$
  - $\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$ (DeMorgan's Law)
  - $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ (Distributive Law)

  - $x_1 = x_4 \vee x_5$
  - $\Leftrightarrow (x_1 \vee (\overline{x_4 \vee x_5})) \wedge (\overline{x_1} \vee (x_4 \vee x_5))$
  - $\Leftrightarrow (x_1 \vee (\overline{x_4} \wedge \overline{x_5})) \wedge (\overline{x_1} \vee (x_4 \vee x_5))$
  - $\Leftrightarrow (x_1 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_5}) \wedge (\overline{x_1} \vee x_4 \vee x_5)$

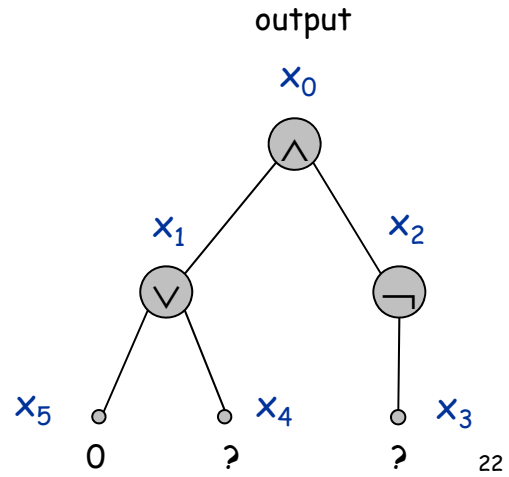


- Hard-coded input values and output value.
  - $x_5 = 0 \Rightarrow$ add 1 clause: $\overline{x_5}$
  - $x_0 = 1 \Rightarrow$ add 1 clause: x_0

- Turn clauses of length 1 or 2 into clauses of length 3.·
  - Create four new variables z_1, z_2, z_3, and z_4.
  - Add 8 clauses to force $z_1 = z_2 =$ false:

$$(\overline{z_1} \vee z_3 \vee z_4),\ (\overline{z_1} \vee z_3 \vee \overline{z_4}),\ (\overline{z_1} \vee \overline{z_3} \vee z_4),\ (\overline{z_1} \vee \overline{z_3} \vee \overline{z_4})$$
$$(\overline{z_2} \vee z_3 \vee z_4),\ (\overline{z_2} \vee z_3 \vee \overline{z_4}),\ (\overline{z_2} \vee \overline{z_3} \vee z_4),\ (\overline{z_2} \vee \overline{z_3} \vee \overline{z_4})$$

  - Replace any clause with a single term (t_i) with ($t_i \vee z_1 \vee z_2$).
  - Replace any clause with two terms ($t_i \vee t_j$) with ($t_i \vee t_j \vee z_1$).



3-satisfiability is NP-complete

Lemma. Φ is satisfiable iff the inputs of K can be set so that it outputs 1.

Pf.

$\Leftarrow$

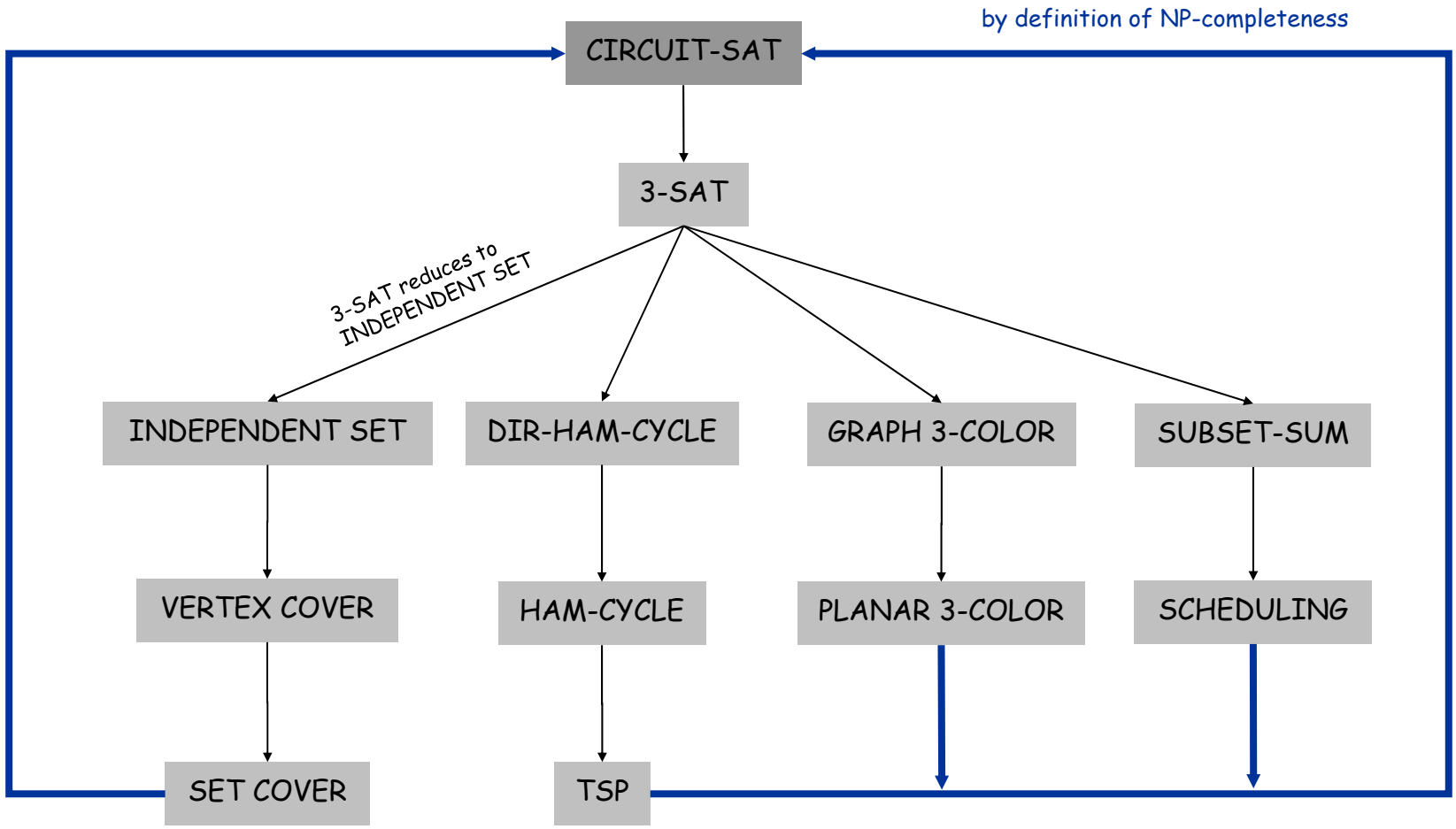
- Suppose there are inputs of K that make it output 1.
- Can propagate input values to create values at all nodes of K.
- This set of values satisfies Φ.

$\Rightarrow$

- Suppose Φ is satisfiable.
- We claim that the set of values corresponding to the circuit inputs constitutes a way to make circuit K output 1.
- The 3-SAT clauses were designed to ensure that the values assigned to all node in K exactly match what the circuit would compute for these nodes. ·

NP-Completeness

Observation. All problems below are NP-complete and polynomial reduce to one another!



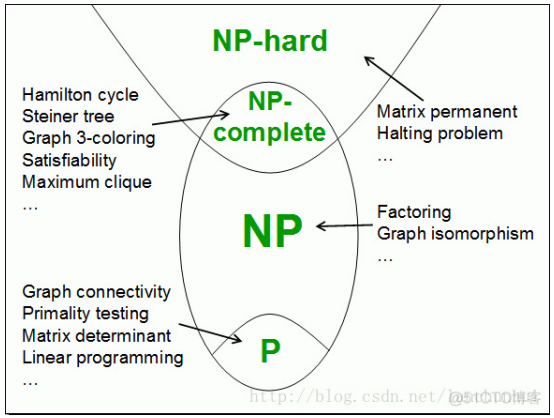
Some NP-Complete Problems

Six basic genres of NP-complete problems and paradigmatic examples.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Practice. Most NP problems are either known to be in P or NP-complete.

Notable exceptions. Factoring, graph isomorphism, Nash equilibrium.



Extent and Impact of NP-Completeness

Extent of NP-completeness. [Papadimitriou 1995]

- Prime intellectual export of CS to other disciplines.
- 6,000 citations per year (title, abstract, keywords).
  - more than "compiler", "operating system", "database"
- Broad applicability and classification power.
- "Captures vast domains of computational, scientific, mathematical endeavors, and seems to roughly delimit what mathematicians and scientists had been aspiring to compute feasibly."

NP-completeness can guide scientific inquiry.

- 1926: Ising introduces simple model for phase transitions.
- 1944: Onsager solves 2D case in tour de force.
- 19xx: Feynman and other top minds seek 3D solution.
- 2000: Istrail proves 3D problem NP-complete.

More Hard Computational Problems

Aerospace engineering: optimal mesh partitioning for finite elements.
Biology: protein folding.
Chemical engineering: heat exchanger network synthesis.
Civil engineering: equilibrium of urban traffic flow.
Economics: computation of arbitrage in financial markets with friction.
Electrical engineering: VLSI layout.
Environmental engineering: optimal placement of contaminant sensors.
Financial engineering: find minimum risk portfolio of given return.
Game theory: find Nash equilibrium that maximizes social welfare.
Genomics: phylogeny reconstruction.
Mechanical engineering: structure of turbulence in sheared flows.
Medicine: reconstructing 3-D shape from biplane angiocardiogram.
Operations research: optimal resource allocation.
Physics: partition function of 3-D Ising model in statistical mechanics.
Politics: Shapley-Shubik voting power.
Pop culture: Minesweeper consistency.
Statistics: optimal experimental design.

8.9 co-NP and the Asymmetry of NP

Asymmetry of NP

Asymmetry of NP. We only need to have short proofs of `yes` instances.

Ex 1. SAT vs. TAUTOLOGY.

- Can prove a CNF formula is satisfiable by giving such an assignment.
- How could we prove that a formula is not satisfiable?

Ex 2. HAM-CYCLE vs. NO-HAM-CYCLE.

- Can prove a graph is Hamiltonian by giving such a Hamiltonian cycle.
- How could we prove that a graph is not Hamiltonian?

Remark. SAT is NP-complete and SAT $\equiv_P$ TAUTOLOGY, but how do we classify TAUTOLOGY?

↑

not even known to be in NP

NP and co-NP

NP. Decision problems for which there is a poly-time certifier.

Ex. SAT, HAM-CYCLE, COMPOSITES.

Def. Given a decision problem X, its complement $\overline{X}$ is the same problem with the `yes` and `no` answers reverse.

Ex. $\overline{X}$ = { 0, 1, 4, 6, 8, 9, 10, 12, 14, 15, … }
X = { 2, 3, 5, 7, 11, 13, 17, 23, 29, … }

co-NP. Complements of decision problems in NP.

Ex. TAUTOLOGY, NO-HAM-CYCLE, PRIMES.

NP = co-NP ?

Fundamental question. Does NP = co-NP?

- Do `yes` instances have succinct certificates iff `no` instances do?
- Consensus opinion: no.

Theorem. If NP ≠ co-NP, then P ≠ NP.

Pf idea.

- P is closed under complementation.
- If P = NP, then NP is closed under complementation.
- In other words, NP = co-NP.
- This is the contrapositive of the theorem.

Good Characterizations

Good characterization. [Edmonds 1965] NP ∩ co-NP.

- If problem X is in both NP and co-NP, then:
  - for `yes` instance, there is a succinct certificate
  - for `no` instance, there is a succinct disqualifier
- Provides conceptual leverage for reasoning about a problem.

Ex. Given a bipartite graph, is there a perfect matching.

- If yes, can exhibit a perfect matching.
- If no, can exhibit a set of nodes S such that $|N(S)| < |S|$.

Good Characterizations

Observation. $P \subseteq NP \cap co\text{-}NP$.

- Proof of max-flow min-cut theorem led to stronger result that max-flow and min-cut are in P.
- Sometimes finding a good characterization seems easier than finding an efficient algorithm.

Fundamental open question. Does $P = NP \cap co\text{-}NP$?

- Mixed opinions.
- Many examples where problem found to have a non-trivial good characterization, but only years later discovered to be in P.
  - linear programming [Khachiyan, 1979]
  - primality testing [Agrawal-Kayal-Saxena, 2002]

Fact. Factoring is in $NP \cap co\text{-}NP$, but not known to be in P.

↑
if poly-time algorithm for factoring, can break RSA cryptosystem

PRIMES is in NP ∩ co-NP

Theorem. PRIMES is in NP ∩ co-NP.

Pf. We already know that PRIMES is in co-NP, so it suffices to prove that PRIMES is in NP.

Pratt's Theorem. An odd integer s is prime iff there exists an integer $1 < t < s$ s.t.

$$t^{s-1} \equiv 1 \pmod{s}$$
$$t^{(s-1)/p} \neq 1 \pmod{s}$$

for all prime divisors p of s-1

Input. s = 437,677

Certificate. t = 17, $2^2 \times 3 \times 36{,}473$

prime factorization of s-1
also need a recursive certificate
to assert that 3 and 36,473 are prime

Certifier.

- Check $s-1 = 2 \times 2 \times 3 \times 36{,}473$.
- Check $17^{s-1} = 1 \pmod{s}$.
- Check $17^{(s-1)/2} \equiv 437{,}676 \pmod{s}$.
- Check $17^{(s-1)/3} \equiv 329{,}415 \pmod{s}$.
- Check $17^{(s-1)/36{,}473} \equiv 305{,}452 \pmod{s}$.

use repeated squaring

FACTOR is in NP ∩ co-NP

FACTORIZE. Given an integer x, find its prime factorization.
FACTOR. Given two integers x and y, does x have a nontrivial factor less than y?

Theorem. FACTOR $\equiv_P$ FACTORIZE.

Theorem. FACTOR is in NP ∩ co-NP.
Pf.

- Certificate: a factor p of x that is less than y.
- Disqualifier: the prime factorization of x (where each prime factor is less than y), along with a certificate that each factor is prime.

Primality Testing and Factoring

We established: PRIMES $\leq_P$ COMPOSITES $\leq_P$ FACTOR.

Natural question: Does FACTOR $\leq_P$ PRIMES ?

Consensus opinion. No.

State-of-the-art.

- PRIMES is in P. ← proved in 2001
- FACTOR not believed to be in P.

RSA cryptosystem.

- Based on dichotomy between complexity of two problems.
- To use RSA, must generate large primes efficiently.
- To break RSA, suffixes to find efficient factoring algorithm.

Extra Slides

Princeton CS Building, West Wall

Princeton CS Building, West Wall

Character	ASCII	Bits
P	80	1010000
=	61	0111101
N	78	1001110
P	80	1010000
?	63	0111111

Not How To Give a PowerPoint Talk

(commercial break)

A Note on Terminology

Knuth. [SIGACT News 6, January 1974, p. 12 – 18]

Find an adjective x that sounds good in sentences like.

- EUCLIDEAN-TSP is x.
- It is x to decide whether a given graph has a Hamiltonian cycle.
- It is unknown whether FACTOR is an x problem.

Note: x does not necessarily imply that a problem is in NP, just that every problem in NP polynomial reduces to x.

A Note on Terminology

Knuth's original suggestions.

- Hard.
- Tough.
- Herculean.
- Formidable.
- Arduous.

but Hercules known for strength not time

Some English word write-ins.

- Impractical.
- Bad.
- Heavy.
- Tricky.
- Intricate.
- Prodigious.
- Difficult.
- Intractable.
- Costly.
- Obdurate.
- Obstinate.
- Exorbitant.
- Interminable.

A Note on Terminology

Hard-boiled. [Ken Steiglitz] In honor of Cook.

Hard-ass. [Al Meyer] Hard as satisfiability.

Sisyphean. [Bob Floyd] Problem of Sisyphus was time-consuming.

but Sisyphus never finished his task

Ulyssean. [Don Knuth] Ulysses was known for his persistence.

and finished!

A Note on Terminology: Made-Up Words

Supersat. [Al Meyer] Greater than or equal to satisfiability.

Polychronious. [Ed Reingold] Enduringly long; chronic.

like today's lecture

PET. [Shen Lin] Probably exponential time.

depending on P=NP conjecture: provably exponential time, or previously exponential time

GNP. [Al Meyer] Greater than or equal to NP in difficulty.

costing more than GNP to resolve

A Note on Terminology: Consensus

NP-complete. A problem in NP such that every problem in NP polynomial reduces to it.

NP-hard. [Bell Labs, Steve Cook, Ron Rivest, Sartaj Sahni]
A decision problem such that every problem in NP reduces to it.

not necessarily in NP

NP-hard search problem. A problem such that every problem in NP reduces to it.

not necessarily a yes/no problem

"creative research workers are as full of ideas for new terminology as they are empty of enthusiasm for adopting it." -*Don Knuth*